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Some integrals involving associated Legendre functions and Gegenbauer polynomials

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Abstract. We present evaluations of integrals involving associated Legendre functions and Gegenbauer polynomials. The calculations are performed using some properties of the generalised hypergeometric series.

In our recent work on meson spectroscopy, we have encountered an integral involving associated Legendre functions (Mita and Laursen 1981). To our knowledge, this particular integral is not found in the literature. The intention of this note is to present the evaluation of the integral in a generalised form for the convenience of theoretical physicists. The method of calculation will be applied also to a similar integral involving Gegenbauer polynomials.

The integral we consider is of the form

$$L(m, n; p) = \int_{-1}^{1} \frac{[P_n^m(x)]^2}{(1-x^2)^{p+1}} \mathrm{d}x \tag{1}$$

where m and n are non-negative integers and $p+1 \leqslant m \leqslant n$. When $p=0$ and $p=-1$, the integral is given in any standard integral table (Gradshteyn and Ryzhik 1965, Bateman 1953). Here, we show an evaluation of the above integral for any integer p employing some properties of the hypergeometric series.

Since

$$P_n^m(x) = \left(-\frac{1}{2}\right)^m \frac{(n+m)!}{(n-m)!\,m!} (1-x^2)^{m/2} {}_2F_1\left[\begin{matrix} m-n, m+n+1 \\ m+1 \end{matrix}; \frac{1-x}{2}\right], \tag{2}$$

we have

$$L(m, n; p) = \frac{1}{2^{2m}} \left(\frac{(n+m)!}{(n-m)!\,m!}\right)^2 \int_{-1}^{1} \left({}_2F_1\left[\begin{matrix} m-n, m+n+1 \\ m+1 \end{matrix}; \frac{1-x}{2}\right]\right)^2$$
$$\times (1-x^2)^{m-p-1} \mathrm{d}x. \tag{3}$$

The hypergeometric series satisfies (Bailey 1964, pp 97, 86)

$${}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix}; x\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}b \\ \frac{1}{2}(a+b+1) \end{matrix}; 4x(1-x)\right] \tag{4}$$

and

$$\left({}_2F_1\left[\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; x\right]\right)^2 = {}_3F_2\left[\begin{matrix} 2a, 2b, a+b \\ 2a, 2b, a+b+\frac{1}{2} \end{matrix}; x\right]. \tag{5}$$

Using these identities, we further rewrite

$$L(m,n;p)=\frac{1}{2^{2m}}\left(\frac{(n+m)!}{(n-m)!\,m!}\right)^2\int_{-1}^{1}{}_3F_2\left[\begin{matrix}m-n,\,m+n+1,\,m+\frac{1}{2}\\2m+1,\,m+1\end{matrix};\;1-x^2\right]$$
$$\times(1-x^2)^{m-p-1}\,\mathrm{d}x$$
$$=\frac{1}{2^{2m}}\left(\frac{(n+m)!}{(n-m)!\,m!}\right)^2\sum_{j=0}^{n-m}\frac{(m-n)_j(m+n+1)_j(m+\frac{1}{2})_j}{(2m+1)_j(m+1)_j j!}$$
$$\times\int_{-1}^{1}(1-x^2)^{j+m-p-1}\,\mathrm{d}x. \tag{6}$$

Now the integration can be performed trivially and we obtain

$$L(m,n;p)=\frac{(m-p-1)!}{2^{m+p}(2m-2p-1)!!}\left(\frac{(n+m)!}{(n-m)!\,m!}\right)^2$$
$$\times{}_4F_3\left[\begin{matrix}m-n,\,m+n+1,\,m+\frac{1}{2},\,m-p\\2m+1,\,m+1,\,m-p+\frac{1}{2}\end{matrix};\;1\right]. \tag{7}$$

In particular, the 'diagonal' element is given by

$$L(m,m;p)=\frac{(m-p-1)!}{2^{m+p}(2m-2p-1)!!}\left(\frac{(2m)!}{m!}\right)^2. \tag{8}$$

One can present the above result in a more convenient form for small p. Since the Saalschützian ${}_4F_3$ satisfies the Whipple transformation (Bailey 1964, p 56) (in the Wilson (1980) symmetric form)†

$${}_4F_3\left[\begin{matrix}-n,\,n+a+b+c+d-1,\,a+t,\,a-t\\a+b,\,a+c,\,a+d\end{matrix};\;1\right]$$
$$=\frac{(b+c)_n(b+d)_n}{(a+c)_n(a+d)_n}\,{}_4F_3\left[\begin{matrix}-n,\,n+a+b+c+d-1,\,b+t,\,b-t\\b+a,\,b+c,\,b+d\end{matrix};\;1\right], \tag{9}$$

we may rewrite equation (7) as

$$L(m,n;p)=\frac{1}{2^{2p+1}}\,\frac{(2m)!}{(2m-2p-1)!}\,\frac{(n+m)!}{(n-m)!}\left(\frac{(m-p-1)!}{m!}\right)^2$$
$$\times{}_4F_3\left[\begin{matrix}m-n,\,m+n+1,\,\frac{1}{2},\,-p\\m-p+\frac{1}{2},\,1,\,m+1\end{matrix};\;1\right] \tag{10}$$

for $p\geqslant 0$, and

$$L(m,n;p)=\frac{1}{2^{2p+1}}\,\frac{(2m)!}{(m!)^2}\,\frac{(n+m)!}{(n-m)!}\,\frac{(m+p)!}{(n+p)!}$$
$$\times\frac{(2n+2p+1)!}{(2m+2p+1)!}\,\frac{(m-p-1)!\,(n-p-1)!}{(2n-2p-1)!}$$
$$\times{}_4F_3\left[\begin{matrix}m-n,\,m+n+1,\,\frac{1}{2},\,p+1\\m+1,\,m+p+\frac{3}{2},\,1\end{matrix};\;1\right] \tag{11}$$

† Incidentally, Minton (1970) examines symmetry properties of the Racah coefficients through the terminating ${}_4F_3$ series.

for $p<0$. Expressed in this fashion, the terminating ${}_4F_3$ now contains only $p+1$ terms when $p\geqslant 0$ and $|p|$ terms when $p<0$. From equations (10) and (11), we obtain some special cases:

$$L(m,n;0)=\frac{1}{m}\frac{(n+m)!}{(n-m)!}, \tag{12}$$

$$L(m,n;1)=\frac{(m-1)(m+1)+n(n+1)}{2m(m-1)(m+1)}\frac{(n+m)!}{(n-m)!}, \tag{13}$$

$$L(m,n;-1)=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, \tag{14}$$

$$L(m,n;-2)=4\frac{(m-1)(m+1)+n(n+1)}{(2n-1)(2n+1)(2n+3)}\frac{(n+m)!}{(n-m)!}. \tag{15}$$

As mentioned before, equations (12) and (14) are given in the integral tables.

As a straightforward generalisation of equation (1), we can also evaluate

$$G(\lambda,n;\mu)=\int_{-1}^{1}[C_n^{\lambda}(x)]^2(1-x^2)^{\lambda+\mu-1/2}\,dx \tag{16}$$

where n is a non-negative integer and $\mathrm{Re}(\lambda+\mu)>-\frac{1}{2}$. Going through the same steps of calculation as before, we obtain

$$G(\lambda,n;\mu)=\frac{B(\lambda+\mu+\frac{1}{2},\frac{1}{2})}{[nB(2\lambda,n)]^2}\,{}_4F_3\left[\begin{matrix}-n,2\lambda+n,\lambda,\lambda+\mu+\frac{1}{2}\\ 2\lambda,\lambda+\frac{1}{2},\lambda+\mu+1\end{matrix};\ 1\right]. \tag{17}$$

When μ is an integer, the Whipple transformation yields

$$G(\lambda,n;\mu)=\frac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}(\lambda+\mu+n)n!\Gamma(\lambda)\Gamma(\lambda-\mu)}$$
$$\times\frac{\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(\lambda-\mu+n)}{\Gamma(\lambda+\frac{1}{2})\Gamma(\lambda+\mu+n)}\,{}_4F_3\left[\begin{matrix}-n,2\lambda+n,\frac{1}{2},-\mu\\ \lambda+\frac{1}{2},\lambda-\mu,1\end{matrix};\ 1\right] \tag{18}$$

for $\mu\geqslant 0$ and

$$G(\lambda,n;\mu)=\frac{\pi\Gamma(2\lambda+n)\Gamma(\lambda+\mu+\frac{1}{2})}{2^{2\lambda-1}n!\Gamma(\lambda)\Gamma(\lambda+\mu+1)\Gamma(\lambda+\frac{1}{2})}\,{}_4F_3\left[\begin{matrix}-n,2\lambda+n,\frac{1}{2},\mu+1\\ \lambda+\mu+1,\lambda+\frac{1}{2},1\end{matrix};\ 1\right] \tag{19}$$

for $\mu<0$. The series contains $\mu+1$ terms when $\mu\geqslant 0$ and $|\mu|$ terms when $\mu<0$. Some special cases are given by

$$G(\lambda,n;0)=\frac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}(\lambda+n)n![\Gamma(\lambda)]^2}, \tag{20}$$

$$G(\lambda,n;1)=\frac{\pi\Gamma(2\lambda+n)[(\lambda-1)(\lambda+\frac{1}{2})+n(\lambda+\frac{1}{2}n)]}{2^{2\lambda-1}(\lambda+n-1)(\lambda+n)(\lambda+n+1)n![\Gamma(\lambda)]^2}, \tag{21}$$

$$G(\lambda,n;-1)=\frac{\pi\Gamma(2\lambda+n)}{2^{2\lambda-1}(\lambda-\frac{1}{2})n![\Gamma(\lambda)]^2}, \tag{22}$$

$$G(\lambda,n;-2)=\frac{\pi\Gamma(2\lambda+n)[(\lambda-1)(\lambda+\frac{1}{2})+n(\lambda+\frac{1}{2}n)]}{2^{2\lambda-1}(\lambda-\frac{1}{2})(\lambda+\frac{1}{2})(\lambda-\frac{3}{2})n![\Gamma(\lambda)]^2}. \tag{23}$$

Of these expressions, equation (20) can be found in the integral tables.

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References

Bailey W N 1964 *Generalized Hypergeometric Series* (New York: Stechert-Hafner)
Bateman Manuscript Project 1953 *Higher Transcendental Functions*, (New York: McGraw-Hill)
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* (New York: Academic)
Minton B M 1970 *J. Math. Phys.* **11** 3061
Mita K and Laursen M L 1981 *Z. Phys.* C **6** 1
Wilson J 1980 *SIAM J. Math. Anal.* **11** 690